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Hamiltonian BRST interactions in Abelian theories

C. Bizdadea[a], M.T. Miautǎ, S.O. Saliu[b]

Faculty of Physics, University of Craiova, 13 A.I. Cuza Str., 1100 Craiova, Romania

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Abstract. Consistent couplings between an Abelian gauge field and three types of matter fields are investigated by means of the Hamiltonian BRST deformation theory based on cohomological techniques. In this manner, scalar electrodynamics, the Stuckelberg theory for Abelian zero- and one-forms, respectively, spinor electrodynamics, are inferred.

1 Introduction

The reformulation of the Lagrangian BRST symmetry [1–5] on cohomological grounds allowed, among others, the study of consistent interactions that can be introduced among fields with gauge freedom without changing the number of gauge symmetries [6–10] with the help of the deformation of the master equation [11] in the framework of the local BRST cohomology [11–16]. This Lagrangian cohomological deformation technique has been successfully applied to many models of interest, like Chern–Simons models, Yang–Mills theories, the Chapline–Manton model, p-forms and chiral p-forms, Einstein's gravity theory, four- and eleven-dimensional supergravity, or BF models [11,17–32].

On the other hand, the Hamiltonian BRST formalism [5,33–37] appears to be the most natural setting for implementing the BRST symmetry in quantum mechanics ([5], Chapter 14). In the meantime, it attracted much attention by providing a strong tool for examining anomalies [38], computing local BRST cohomologies [39], as well as for establishing a proper connection with canonical quantization formalisms, like, for instance, the reduced phase-space or Dirac quantization procedures [40]. Lately, the Hamiltonian BRST approach has been extended to the investigation of consistent interactions that can be added in gauge theories with the help of the deformation technique based on local cohomologies [41–44].

In this paper we investigate the consistent Hamiltonian interactions that can be introduced between an Abelian gauge field and three types of matter fields, namely, the complex scalar, the massless real scalar and Dirac, with the help of cohomological BRST arguments combined with the deformation technique. In each of the three cases under consideration we start from a "free" theory, whose Lagrangian action is equal to the sum of the action of an Abelian gauge field and the one describing one of the matter fields. Every of the "free" systems displays two types of symmetries: a rigid one related to the matter component, that induces a certain conserved current, and the other purely gauge, characteristic to Maxwell's theory. The Hamiltonian BRST symmetry of the "free" models, s, simply decomposes into $s = \delta + \gamma$, with δ the Koszul–Tate differential and γ the exterior derivative along the gauge orbits. Its non-trivial action is essentially due to the first-class constraints of the electromagnetic field. It has been shown in [41–44] that the Hamiltonian problem of introducing consistent interactions in gauge theories can be reformulated as a deformation problem of the BRST charge and BRST-invariant Hamiltonian of a starting "free" theory. Following this line, we prove that the deformed BRST charge consistent at all orders in the deformation parameter can be taking non-vanishing only at order one in the case of all the investigated models. Meanwhile, the first-order deformation of the BRST charge reduces every time to the component of antighost number zero, which is γ-invariant. Further, we solve the equations responsible for the deformation of the BRST-invariant Hamiltonian associated with the "free" systems. Related to the first-order deformation equation written in a local form, we give evidence for its relationship with the conserved currents corresponding to some rigid transformations of the matter fields. On account of this relationship, we can determine the deformed BRST charge and the first-order deformation of the BRST-invariant Hamiltonian. The remaining higher-order equations are then satisfactorily solved, and the deformed BRST-invariant Hamiltonian is completely output in every of the cases under study. It is important to notice that there appear no obstructions regarding the locality of the deformed BRST quantities. Analyzing the resulting interacting models, it follows that we have constructed nothing but scalar electrodynamics, the Stuckelberg theory for Abelian zero- and one-forms and spinor electrodynamics. The matter fields are endowed, as a consequence of their couplings to the Abelian gauge field, with some gauge transformations that can be inferred from the original global ones merely by gauging.

[a] e-mail: bizdadea@central.ucv.ro
[b] e-mail: osaliu@central.ucv.ro

This paper is organized in five sections. Section 2 briefly formulates the analysis of consistent Hamiltonian interactions that can be added to a "free" theory without changing its number of gauge symmetries as a deformation problem of the corresponding BRST charge and BRST-invariant Hamiltonian, finally expressed in terms of the so-called main equations. Based on this, in Sect. 3 we construct the consistent Hamiltonian couplings between an Abelian gauge field and a scalar field on cohomological grounds. As a consequence, we infer scalar electrodynamics in the complex case, respectively, the Stuckelberg model involving zero- and one-forms in the massless real case. Section 4 deals with a similar topic with respect to an Abelian gauge field and a Dirac field, leading to spinor electrodynamics. Section 5 ends the paper with some conclusions.

2 Main equations of the Hamiltonian BRST deformation procedure

We consider a "free" Lagrangian theory, whose action is invariant under some gauge transformations, that can in principle be reducible. At the Lagrangian level, all the information on the gauge structure and reducibility relations is encoded within the solution to the master equation. Moreover, it has been shown that the deformation of the solution to the master equation generates consistent interactions among fields with gauge freedom [5]. At the Hamiltonian level, the gauge structure of a given gauge theory is completely captured by the BRST charge and BRST-invariant Hamiltonian. Similarly to the Lagrangian deformation procedure, we can reformulate the problem of introducing consistent Hamiltonian interactions like a deformation problem of the BRST charge and BRST-invariant Hamiltonian.

If the interactions can be consistently constructed, then the BRST charge of a given "free" theory, Ω_0, can be deformed as

$$\begin{aligned}\Omega_0 \to \Omega &= \Omega_0 + g\int \mathrm{d}^{D-1}x\,\omega_1 + g^2\int \mathrm{d}^{D-1}x\,\omega_2 + O(g^3)\\ &= \Omega_0 + g\Omega_1 + g^2\Omega_2 + O(g^3),\end{aligned}\tag{1}$$

where Ω should satisfy the equation

$$[\Omega,\Omega]=0. \tag{2}$$

Here, the symbol $[,]$ denotes either the Poisson, or the Dirac bracket. If the initial system is purely first class, we need the Poisson bracket; if there are also second-class constraints, then we eliminate them, and work with the Dirac one. Equation (2) splits accordingly with the deformation parameter g as

$$[\Omega_0,\Omega_0]=0, \tag{3}$$

$$2[\Omega_0,\Omega_1]=0, \tag{4}$$

$$2[\Omega_0,\Omega_2]+[\Omega_1,\Omega_1]=0, \tag{5}$$

$$\vdots$$

Obviously, (3) is automatically satisfied. From the remaining equations we deduce the pieces $(\Omega_k)_{k>0}$ on account of the "free" BRST differential. With the deformed BRST charge at hand, we then deform the BRST-invariant Hamiltonian of the "free" theory, $H_{0\mathrm{B}}$, like

$$\begin{aligned}H_{0\mathrm{B}} \to H_{\mathrm{B}} &= H_{0\mathrm{B}} + g\int \mathrm{d}^{D-1}x\,h_1\\ &\quad + g^2\int \mathrm{d}^{D-1}x\,h_2 + O(g^3)\\ &= H_{0\mathrm{B}} + gH_1 + g^2H_2 + O(g^3),\end{aligned}\tag{6}$$

and require that

$$[H_{\mathrm{B}},\Omega]=0. \tag{7}$$

Equation (7) can be analyzed order by order in the deformation parameter g, leading to

$$[H_{0\mathrm{B}},\Omega_0]=0, \tag{8}$$

$$[H_{0\mathrm{B}},\Omega_1]+[H_1,\Omega_0]=0, \tag{9}$$

$$[H_{0\mathrm{B}},\Omega_2]+[H_1,\Omega_1]+[H_2,\Omega_0]=0, \tag{10}$$

$$\vdots$$

Clearly, (8) is again fulfilled, while from the others one can determine the components $(H_k)_{k>0}$ by relying on the BRST symmetry of the "free" model. Equations (3)–(5), etc., and (8)–(10), etc., represent the main equations of our Hamiltonian deformation procedure. They will be solved in the next sections with respect to the models under study by means of some cohomological techniques, specific to the Hamiltonian BRST formalism.

3 Couplings between an Abelian gauge field and a scalar field

Initially, we investigate the consistent Hamiltonian couplings between an Abelian gauge field and a scalar field along the line exposed in the above, and derive the scalar electrodynamics in the complex case, respectively, the Stuckelberg coupling in the massless real case.

3.1 Couplings with a complex scalar field

We start from a "free" Lagrangian action written as the sum between the action of an Abelian gauge field A_μ and that of a complex scalar field $(\varphi,\bar{\varphi})$

$$S_0^{\mathrm{L}}[\varphi,\bar{\varphi},A^\mu] = \int \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu\varphi)\partial^\mu\bar{\varphi} - \mu^2\varphi\bar{\varphi} - V(\varphi\bar{\varphi})\right), \quad (11)$$

where the bar operation represents the complex conjugation, while the Abelian field strength is defined in the usual manner by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

As commonly known, the action (11) is invariant under the one-parameter rigid symmetries (genuinely, only the part corresponding to the scalar field is non-trivially responsible for this global invariance)

$$\Delta\varphi = \mathrm{i}\varphi\xi, \quad \Delta\bar{\varphi} = -\mathrm{i}\bar{\varphi}\xi, \quad (12)$$

leading, via Noether's theorem, to the conservation law

$$\partial_\mu j^\mu = \mathrm{i}\varphi\frac{\delta\mathcal{L}^{(\mathrm{S})}}{\delta\varphi} - \mathrm{i}\bar{\varphi}\frac{\delta\mathcal{L}^{(\mathrm{S})}}{\delta\bar{\varphi}}, \quad (13)$$

giving evidence for the conserved current

$$j^\mu = \mathrm{i}(\bar{\varphi}\partial^\mu\varphi - \varphi\partial^\mu\bar{\varphi}), \quad (14)$$

with

$$\frac{\delta\mathcal{L}^{(\mathrm{S})}}{\delta\varphi} = -\left(\partial_\mu\partial^\mu + \mu^2 + \frac{\partial V}{\partial(\varphi\bar{\varphi})}\right)\bar{\varphi}, \quad (15)$$

$$\frac{\delta\mathcal{L}^{(\mathrm{S})}}{\delta\bar{\varphi}} = -\left(\partial_\mu\partial^\mu + \mu^2 + \frac{\partial V}{\partial(\varphi\bar{\varphi})}\right)\varphi, \quad (16)$$

where $\mathcal{L}^{(\mathrm{S})}$ stands for the Lagrangian density associated with the complex scalar field.

By passing to the canonical analysis of action (11), we find the Abelian first-class constraints and first-class Hamiltonian of the form

$$G_1 \equiv \pi_0 \approx 0, \quad G_2 \equiv -\partial^i\pi_i \approx 0, \quad (17)$$

$$H_0 = \int \mathrm{d}^3x \Big(\frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F^{ij} - A^0\partial^i\pi_i + \pi\bar{\pi} - (\partial_j\varphi)\partial^j\bar{\varphi} + \mu^2\varphi\bar{\varphi} + V(\varphi\bar{\varphi})\Big), \quad (18)$$

where π_μ, π and $\bar{\pi}$ denote the canonical momenta of the fields A^μ, φ, respectively, $\bar{\varphi}$. The BRST charge of this "free" theory is then

$$\Omega_0 = \int \mathrm{d}^3x(\pi_0\eta^1 - (\partial^i\pi_i)\eta^2), \quad (19)$$

where $\eta^1$and η^2 represent the fermionic Hamiltonian ghosts. Their antighosts, to be denoted by $\mathcal{P}_1$, respectively, $\mathcal{P}_2$, are also fermionic. The "free" Hamiltonian BRST symmetry $s\bullet = [\bullet,\Omega_0]$ simply decomposes as

$$s = \delta + \gamma, \quad (20)$$

with δ the Koszul–Tate differential, and γ the exterior longitudinal derivative along the gauge orbits. The Koszul–Tate differential is graded according to the antighost number (antigh, $\mathrm{antigh}(\delta) = -1$), the degree of the exterior longitudinal derivative is named the pure ghost number (pgh, $\mathrm{pgh}(\gamma) = 1$, $\mathrm{pgh}(\delta) = 0$, $\mathrm{antigh}(\gamma) = 0$), while the overall grading of the BRST differential is called the ghost number (gh, $\mathrm{gh}(s) = 1$), and is defined by the difference between the pure ghost number and the antighost number. The degrees of the generators from the BRST complex are valued

$$\mathrm{antigh}(A^\mu) = \mathrm{antigh}(\pi_\mu) = \mathrm{antigh}(\varphi) = \mathrm{antigh}(\bar{\varphi}) = 0, \quad (21)$$

$$\mathrm{antigh}(\pi) = \mathrm{antigh}(\bar{\pi}) = 0, \quad \mathrm{antigh}(\mathcal{P}_a) = 1,\ \mathrm{antigh}(\eta^a) = 0,\ a = 1,2, \quad (22)$$

$$\mathrm{pgh}(A^\mu) = \mathrm{pgh}(\pi_\mu) = \mathrm{pgh}(\varphi) = \mathrm{pgh}(\bar{\varphi}) = \mathrm{pgh}(\pi) = \mathrm{pgh}(\bar{\pi}) = 0, \quad (23)$$

$$\mathrm{pgh}(\mathcal{P}_a) = 0,\ \mathrm{pgh}(\eta^a) = 1,\ a = 1,2. \quad (24)$$

The operators δ and γ act on the BRST generators through the relations

$$\delta A^\mu = 0,\ \delta\pi_\mu = 0,\ \delta\varphi = 0,\ \delta\bar{\varphi} = 0,\ \delta\pi = 0, \quad (25)$$

$$\delta\bar{\pi} = 0,\ \delta\mathcal{P}_1 = -\pi_0,\ \delta\mathcal{P}_2 = \partial^i\pi_i,\ \delta\eta^1 = 0,\ \delta\eta^2 = 0, \quad (26)$$

$$\gamma A^0 = \eta^1,\ \gamma A^i = \partial^i\eta^2,\ \gamma\pi_\mu = 0,\ \gamma\varphi = 0,\ \gamma\bar{\varphi} = 0,\ \gamma\pi = 0, \quad (27)$$

$$\gamma\bar{\pi} = 0,\ \gamma\mathcal{P}_1 = 0,\ \gamma\mathcal{P}_2 = 0,\ \gamma\eta^1 = 0,\ \gamma\eta^2 = 0, \quad (28)$$

that will be used in the sequel at the deformation procedure.

Next, we solve the equations (4)–(5), etc., and (9)–(10), etc., that govern the Hamiltonian deformation. Taking into account the expression (2), the local form of (4) holds if and only if ω_1 is an s-co-cycle modulo the spatial part of the space-time derivative, $\tilde{\mathrm{d}} = \mathrm{d}x^i\partial_i$, hence if and only if

$$s\omega_1 = \partial_k\sigma^k, \quad (29)$$

for some σ^k. In order to solve (29) we expand ω_1 according to the antighost number

$$\omega_1 = \overset{(0)}{\omega}_1 + \overset{(1)}{\omega}_1 + \cdots + \overset{(J)}{\omega}_1, \quad (30)$$

where the last term can be assumed to be annihilated by γ. As

$$\mathrm{antigh}(\overset{(J)}{\omega}_1) = J \text{ and } \mathrm{gh}(\overset{(J)}{\omega}_1) = 1,$$

we find the result that

$$\mathrm{pgh}(\overset{(J)}{\omega}_1) = J + 1,$$

so we can represent $\overset{(J)}{\omega}_1$ in the form

$$\overset{(J)}{\omega}_1 = \mu_J(\eta^2)^{J+1}.$$

(The ghost η^1 does not come into discussion as it is trivial in the cohomology of γ: $\gamma A^0 = \eta^1$, $\gamma\eta^1 = 0$.) Due to the fermionic character of η^2, this term is non-vanishing if and only if $J = 0$, such that

$$\omega_1 = \overset{(0)}{\omega}_1 = \mu_0 \eta^2. \tag{31}$$

With this choice, it is easy to check that the γ-invariant coefficient μ_0 should satisfy the conditions $\mathrm{antigh}(\mu_0) = 0$, $\mathrm{pgh}(\mu_0) = 0$, $\gamma\mu_0 = 0$. From (25)–(28) the result is obtained that μ_0 can depend on $(\varphi, \bar{\varphi})$ and $(\pi, \bar{\pi})$, so $\mu_0 = \mu_0(\varphi, \bar{\varphi}, \pi, \bar{\pi})$. In this way, the first-order deformation of the BRST charge, determined up to μ_0, is given by

$$\Omega_1 = \int \mathrm{d}^3x \, \mu_0(\varphi, \bar{\varphi}, \pi, \bar{\pi})\eta^2. \tag{32}$$

By direct computation we then obtain that $[\Omega_1, \Omega_1] = 0$, no matter what $\mu_0(\varphi, \bar{\varphi}, \pi, \bar{\pi})$ we take. The second-order deformation equation of the BRST charge, (5), is thus satisfied for $\Omega_2 = 0$, such that the corresponding higher-order deformations can be taken as $\Omega_3 = \Omega_4 = \cdots = 0$. Consequently, the overall deformed BRST charge takes the form

$$\Omega = \int \mathrm{d}^3x(\pi_0\eta^1 - (\partial^i\pi_i - g\mu_0(\varphi, \bar{\varphi}, \pi, \bar{\pi}))\eta^2). \tag{33}$$

At this point, we investigate the deformation of the BRST-invariant Hamiltonian, described by (9), (10), etc., where the BRST-invariant Hamiltonian of the free theory reads

$$H_{0\mathrm{B}} = H_0 + \int \mathrm{d}^3x \, \eta^1 \mathcal{P}_2. \tag{34}$$

From (18), (32) and (34) we see that the first term in (9) can be written as

$$\begin{aligned}
[H_{0\mathrm{B}}, \Omega_1] = \int \mathrm{d}^3x \Big(& -(\pi\bar{u} + \bar{\pi}u)\eta^2 - \eta^1\mu_0 \\
& + \left(\left(\partial_j\partial^j\varphi + \mu^2\varphi + \frac{\partial V}{\partial(\varphi\bar{\varphi})}\varphi \right) \bar{v} \right. \\
& \left. + \left(\partial_j\partial^j\bar{\varphi} + \mu^2\bar{\varphi} + \frac{\partial V}{\partial(\varphi\bar{\varphi})}\bar{\varphi} \right) v \right) \eta^2 \Big) \\
= \int & \mathrm{d}^3x \, \lambda,
\end{aligned} \tag{35}$$

so that the local form of (9) leads to

$$sh_1 + \lambda = \partial^i n_i, \tag{36}$$

for some n_i. In the above we used the notations $\bar{u}(x) = \int \mathrm{d}^3y(\delta\mu_0(x^0, \boldsymbol{y})/\delta\bar{\varphi}(x))$, $u(x) = \int \mathrm{d}^3y(\delta\mu_0(x^0, \boldsymbol{y})/\delta\varphi(x))$, $\bar{v}(x) = \int \mathrm{d}^3y(\delta\mu_0(x^0, \boldsymbol{y})/\delta\bar{\pi}(x))$, $v(x) = \int \mathrm{d}^3y(\delta\mu_0(x^0, \boldsymbol{y})/\delta\pi(x))$. As the term $-\eta^1\mu_0$ from λ does not contain spatial derivatives, it should be compensated by a similar term of opposite sign in sh_1. This can be achieved if and only if

$$h_1 = \mu_0 A^0 + \alpha, \tag{37}$$

where α should depend on A^i in order to produce a term containing spatial derivatives through its Poisson bracket with the second term in (19). In the meantime, α involves no ghosts or antighosts because otherwise we would enlarge $[H_1, \Omega_0]$ with pieces that are not present in $[H_{0\mathrm{B}}, \Omega_1]$. These considerations further give

$$[H_1, \Omega_0] = \int \mathrm{d}^3x(\eta^1\mu_0 + a_i\partial^i\eta^2), \tag{38}$$

which combined with (35), lead to the concrete form of (36) as

$$\begin{aligned}
& \left(\left(\partial_j\partial^j\varphi + \mu^2\varphi + \frac{\partial V}{\partial(\varphi\bar{\varphi})}\varphi \right) \bar{v} \right. \\
& \quad \left. + \left(\partial_j\partial^j\bar{\varphi} + \mu^2\bar{\varphi} + \frac{\partial V}{\partial(\varphi\bar{\varphi})}\bar{\varphi} \right) v - \pi\bar{u} - \bar{\pi}u \right) \eta^2 \\
& + a_i\partial^i\eta^2 = \partial^i n_i,
\end{aligned} \tag{39}$$

where $a_i(x) = \int \mathrm{d}^3y(\delta\alpha(x^0, \boldsymbol{y})/\delta A^i(x))$. In order to obtain a total derivative in the left-hand side of (39) we must have

$$\begin{aligned}
& \left(\partial_j\partial^j\varphi + \mu^2\varphi + \frac{\partial V}{\partial(\varphi\bar{\varphi})}\varphi \right) \bar{v} + \left(\partial_j\partial^j\bar{\varphi} + \mu^2\bar{\varphi} + \frac{\partial V}{\partial(\varphi\bar{\varphi})}\bar{\varphi} \right) \\
& \times v - \pi\bar{u} - \bar{\pi}u = \partial^i a_i.
\end{aligned} \tag{40}$$

By adding the terms $(\partial_0\partial^0\varphi)\bar{v}$ and $(\partial_0\partial^0\bar{\varphi})v$ to both sides of the above equation, we arrive at

$$\begin{aligned}
& -\frac{\delta\mathcal{L}^{(S)}}{\delta\varphi}v - \frac{\delta\mathcal{L}^{(\mathrm{S})}}{\delta\bar{\varphi}}\bar{v} \\
& = \pi\bar{u} + \bar{\pi}u + (\partial_0\partial^0\varphi)\bar{v} + (\partial_0\partial^0\bar{\varphi})v + \partial^i a_i.
\end{aligned} \tag{41}$$

The left-hand side of (41) represents nothing but the variation of the Lagrangian density of the complex scalar field under the rigid transformations

$$\begin{aligned}
\Delta\varphi(x) &= -\int \mathrm{d}^3y \frac{\delta\mu_0(x^0, \boldsymbol{y})}{\delta\pi(x)}\xi, \\
\Delta\bar{\varphi}(x) &= -\int \mathrm{d}^3y \frac{\delta\mu_0(x^0, \boldsymbol{y})}{\delta\bar{\pi}(x)}\xi.
\end{aligned} \tag{42}$$

On the other hand, by identifying the above global variations with the rigid one-parameter transformations (12), we get the equations

$$\begin{aligned}
\int \mathrm{d}^3y \frac{\delta\mu_0(x^0, \boldsymbol{y})}{\delta\pi(x)} &= -\mathrm{i}\varphi(x), \\
\int \mathrm{d}^3y \frac{\delta\mu_0(x^0, \boldsymbol{y})}{\delta\bar{\pi}(x)} &= \mathrm{i}\bar{\varphi}(x),
\end{aligned} \tag{43}$$

whose solution outputs the unknown function μ_0 of the type

$$\mu_0(y) = \mathrm{i}(\bar{\varphi}\bar{\pi} - \varphi\pi)(y). \tag{44}$$

Inserting (44) in (41), and taking into account (13), we find that $\int \mathrm{d}^3y(\delta\alpha(x^0, \boldsymbol{y})/\delta A^i(x)) = \mathrm{i}(\bar{\varphi}\partial_i\varphi - \varphi\partial_i\bar{\varphi})(x)$, which yields

$$\alpha(y) = (\mathrm{i}(\bar{\varphi}\partial_i\varphi - \varphi\partial_i\bar{\varphi})A^i)(y). \tag{45}$$

In this manner, we have completely determined the first-order deformation of the BRST-invariant Hamiltonian and BRST charge:

$$H_1 = \mathrm{i}\int \mathrm{d}^3x((\bar{\varphi}\bar{\pi} - \varphi\pi)A^0 + (\bar{\varphi}\partial_i\varphi - \varphi\partial_i\bar{\varphi})A^i), \quad (46)$$

$$\Omega_1 = \mathrm{i}\int \mathrm{d}^3x(\bar{\varphi}\bar{\pi} - \varphi\pi)\eta^2. \quad (47)$$

Next, we approach the equation responsible for the second-order deformation of the BRST-invariant Hamiltonian, (10). In view of this, we remark that the first term is vanishing as $\Omega_2 = 0$, while the second one is equal to

$$[H_1, \Omega_1] = -2\int \mathrm{d}^3x(\partial_i(\varphi\bar{\varphi}A^i))\eta^2 = \int \mathrm{d}^3x\,\rho. \quad (48)$$

Consequently, (10) written in a local form becomes

$$sh_2 + \rho = \partial_i k^i, \quad (49)$$

whose solution reads

$$h_2 = -\varphi\bar{\varphi}A^iA_i, \quad (50)$$

so that

$$sh_2 + \rho = \partial_i(-2\varphi\bar{\varphi}A^i\eta^2). \quad (51)$$

Passing now to the third-order equation, $[H_{0\mathrm{B}}, \Omega_3] + [H_1, \Omega_2] + [H_2, \Omega_1] + [H_3, \Omega_0] = 0$, we remark that the first two terms vanish as $\Omega_2 = \Omega_3 = 0$, while by direct computation we obtain

$$[H_2, \Omega_1] = 0. \quad (52)$$

Thus, we can safely take the third-order deformation piece in the BRST-invariant Hamiltonian to be equal to zero, $H_3 = 0$, and, moreover, it turns out that all higher-order deformation equations are fulfilled for

$$H_4 = H_5 = \cdots = 0. \quad (53)$$

Synthesizing the results deduced so far, we find that the complete deformations of the BRST charge and BRST-invariant Hamiltonian associated with the "free" system under discussion are

$$\Omega = \int \mathrm{d}^3x(\pi_0\eta^1 - (\partial^i\pi_i - \mathrm{i}g(\bar{\varphi}\bar{\pi} - \varphi\pi))\eta^2), \quad (54)$$

respectively,

$$H_{\mathrm{B}} = \int \mathrm{d}^3x\Big(\frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F^{ij} - A^0\partial^i\pi_i + \pi\bar{\pi} - (\partial_j\varphi)(\partial^j\bar{\varphi}) + \mu^2\varphi\bar{\varphi} + V(\varphi\bar{\varphi}) + \mathrm{i}g(\bar{\varphi}\bar{\pi} - \varphi\pi)A^0 + \mathrm{i}g(\bar{\varphi}\partial_i\varphi - \varphi\partial_i\bar{\varphi})A^i - g^2\varphi\bar{\varphi}A^iA_i + \eta^1\mathcal{P}_2\Big). \quad (55)$$

Now, we are in the position to analyze the resulting deformed theory. From the pieces present in Ω that are linear in the ghosts we observe that the resulting model displays at the Hamiltonian level the same primary first-class constraint like the initial system (the former constraint in (17)), but the secondary one as a result of the deformation process has turned into

$$\gamma_2 \equiv -\partial^i\pi_i + \mathrm{i}g(\bar{\varphi}\bar{\pi} - \varphi\pi) \approx 0, \quad (56)$$

such that these first-class constraints are still Abelian. Examining the terms that contain neither ghosts nor antighosts in (55), we notice that the first-class Hamiltonian of the interacting theory reads

$$H = \int \mathrm{d}^3x\Big(\frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F^{ij} - A^0(\partial^i\pi_i - \mathrm{i}g(\bar{\varphi}\bar{\pi} - \varphi\pi)) + \pi\bar{\pi} - (D_j\varphi)(\overline{D^j\varphi}) + \mu^2\varphi\bar{\varphi} + V(\varphi\bar{\varphi})\Big), \quad (57)$$

where the spatial part of the covariant derivative is defined through

$$D_i = \partial_i + \mathrm{i}gA_i. \quad (58)$$

The Lagrangian setting of the deformed system can be derived by successively passing to the extended and total formalisms, which finally yields the Lagrangian action

$$S^{\mathrm{L}}[\varphi, \bar{\varphi}, A^\mu] = \int \mathrm{d}^4x\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)(\overline{D^\mu\varphi}) - \mu^2\varphi\bar{\varphi} - V(\varphi\bar{\varphi})\Big), \quad (59)$$

subject to the gauge transformations

$$\delta_\epsilon A^\mu = \partial^\mu\epsilon, \quad \delta_\epsilon\varphi = \mathrm{i}g\varphi\epsilon, \quad \delta_\epsilon\bar{\varphi} = -\mathrm{i}g\bar{\varphi}\epsilon, \quad (60)$$

with the covariant derivative given by

$$D_\mu = \partial_\mu + \mathrm{i}gA_\mu. \quad (61)$$

We remark that the complex scalar field, that initially possessed only the rigid invariances (12), becomes endowed now with the gauge invariances in (60), that can be directly obtained from the rigid ones merely by gauging, and, moreover, have a typical form of gauge invariances for matter fields. It appears to be clear that the resulting interacting theory describes, at both the Hamiltonian and the Lagrangian level, nothing but the coupling between an Abelian gauge field and a complex scalar field, which is known as scalar electrodynamics.

3.2 Couplings with a massless real scalar field

In the sequel we apply the Hamiltonian deformation scheme to a free theory involving a massless real scalar field φ and an Abelian gauge field A^μ, and arrive precisely at a model underlying the Stuckelberg coupling between them. The Lagrangian action of this free system is

$$S_0^{\prime\mathrm{L}}[\varphi, A^\mu] = \int \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi)\right), \quad (62)$$

and possesses the global shift symmetry

$$\Delta\varphi = \xi, \quad (63)$$

due essentially to the presence of the real scalar field, which leads to the conservation law

$$\partial_\mu j^\mu = \frac{\delta\mathcal{L}^{(\mathrm{SR})}}{\delta\varphi}, \quad (64)$$

which reveals the conserved current

$$j^\mu = -\partial^\mu\varphi, \quad (65)$$

where

$$\frac{\delta\mathcal{L}^{(\mathrm{SR})}}{\delta\varphi} = -\partial_\mu\partial^\mu\varphi, \quad (66)$$

and $\mathcal{L}^{(\mathrm{SR})}$ denotes the Lagrangian density of the real scalar field. From the canonical analysis of this theory we get the Abelian first-class constraints (17) and the first-class Hamiltonian

$$H_0' = \int \mathrm{d}^3x \left(\frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F^{ij} - A^0\partial^i\pi_i + \frac{1}{2}\pi^2 - \frac{1}{2}(\partial_i\varphi)(\partial^i\varphi)\right), \quad (67)$$

where π is the momentum conjugated to φ. The BRST analysis is exactly the same like that performed for the previous model, and relies on the formulas (19)–(28), from which any reference to the pair $(\bar\varphi, \bar\pi)$ should be discarded. At this point, we have all the elements required for the development of the Hamiltonian deformation scheme.

The consistent deformations of the free BRST charge (19) demand, as we have seen, finding the non-trivial solutions to (4), (5), etc. The first-order deformation equation takes the local form (29). Reasoning like above, we develop ω_1 according to the antighost number (see (30)), and conclude that it reduces to the first component

$$\omega_1 = \stackrel{(0)}{\omega}_1 = \mu_0'(\varphi, \pi)\eta^2, \quad (68)$$

where the function $\mu_0'(\varphi, \pi)$ is unknown and γ-invariant, such that the deformed BRST charge takes the form (35), with $\mu_0'(\varphi, \pi)$ instead of $\mu_0(\varphi, \bar\varphi, \pi, \bar\pi)$.

Investigating in the sequel the deformation of the BRST-invariant Hamiltonian (34) (with H_0 replaced by H_0') at the first-order level, described by (9), it follows, with the help of the relation (68), that

$$[H_{0\mathrm{B}}, \Omega_1] = \int \mathrm{d}^3x((-\pi u' + (\partial_j\partial^j\varphi)v')\eta^2 - \eta^1\mu_0') = \int \mathrm{d}^3x\, \lambda', \quad (69)$$

where $u'(x) = \int \mathrm{d}^3y(\delta\mu_0'(x^0, \boldsymbol{y})/\delta\varphi(x))$ and $v'(x) = \int \mathrm{d}^3y(\delta\mu_0'(x^0, \boldsymbol{y})/\delta\pi(x))$, hence the local form of (9) can be written as

$$sh_1 + \lambda' = \partial^i n_i'. \quad (70)$$

Now, we take h_1 as

$$h_1 = \mu_0' A^0 + \alpha', \quad (71)$$

in order to discard the term $\eta^1\mu_0'$ from the left-hand side of (9), where α' has both the antighost and pure ghost numbers equal to zero and depends in a non-trivial way of A^i for the same reason as before. After some computation, we deduce that

$$[H_1, \Omega_0] = \int \mathrm{d}^3x(\eta^1\mu_0' + a_i'\partial^i\eta^2), \quad (72)$$

with $a_i'(x) = \int \mathrm{d}^3y(\delta\alpha'(x^0, \boldsymbol{y})/\delta A^i(x))$. Therefore, (70) becomes

$$(-\pi u' + (\partial_j\partial^j\varphi)v')\eta^2 + a_i'\partial^i\eta^2 = \partial^i n_i', \quad (73)$$

and it is satisfied if we impose

$$-\pi u' + (\partial_j\partial^j\varphi)v' = \partial^i a_i'. \quad (74)$$

If we add the term $(\partial_0\partial^0\varphi)v'$ to both sides of the last equation, we find the relation

$$\frac{\delta\mathcal{L}^{(\mathrm{SR})}}{\delta\varphi}v' = -(\pi u' + (\partial_0\partial^0\varphi)v' + \partial^i a_i'), \quad (75)$$

whose left-hand side signifies the variation of the Lagrangian density of the real scalar field under the one-parameter rigid transformations

$$\Delta\varphi(x) = \int \mathrm{d}^3y \frac{\delta\mu_0'(x^0, \boldsymbol{y})}{\delta\pi(x)}\xi. \quad (76)$$

Then by identifying (76) with the global shift invariance (63), characteristic for the real scalar field, we are led to the equation

$$\int \mathrm{d}^3y \frac{\delta\mu_0'(x^0, \boldsymbol{y})}{\delta\pi(x)} = 1, \quad (77)$$

possessing the solution

$$\mu_0'(y) = \pi(y), \quad (78)$$

that substituted in (75) reveals the equation $\int \mathrm{d}^3y(\delta\alpha'(x^0, \boldsymbol{y})/\delta A^i(x)) = \partial_i\varphi(x)$, clearly leading to

$$\alpha'(y) = A^i\partial_i\varphi(y). \quad (79)$$

So far, we have generated the first-order deformation of the BRST-invariant Hamiltonian and BRST charge related to the free model under consideration:

$$H_1 = \int d^3x (\pi A^0 + A^i \partial_i \varphi), \tag{80}$$

$$\Omega_1 = \int d^3x \, \pi \eta^2. \tag{81}$$

Further, we remark that the first term in the second-order deformation equation of the BRST-invariant Hamiltonian, (10), is equal to zero due to $\Omega_2 = 0$; the second piece is found to be

$$[H_1, \Omega_1] = -\int d^3x (\partial_i A^i) \eta^2 = \int d^3x \, \rho', \tag{82}$$

hence (10) is equivalent to $sh_2 + \rho' = \partial_i k'^i$, and allows us to write

$$h_2 = -\frac{1}{2} A^i A_i, \tag{83}$$

so that

$$sh_2 + \rho' = \partial_i (-A^i \eta^2). \tag{84}$$

Then it is easy to check that $[H_2, \Omega_1] = 0$, which produces $H_3 = 0$, and consequently $H_4 = H_5 = \cdots = 0$.

According to the results obtained until now, we can state that the deformed BRST charge and BRST-invariant Hamiltonian corresponding to the model that describes an Abelian gauge field coupled with a real scalar field, consistent to all orders in the deformation parameter, take the form

$$\Omega = \int d^3x (\pi_0 \eta^1 - (\partial^i \pi_i - g\pi) \eta^2), \tag{85}$$

respectively,

$$\begin{aligned} H_B = \int d^3x \Big(& \frac{1}{2} \pi_i \pi_i + \frac{1}{4} F_{ij} F^{ij} \\ & - A^0 \partial^i \pi_i + \frac{1}{2} \pi^2 - \frac{1}{2} (\partial_i \varphi)(\partial^i \varphi) \\ & + g \pi A^0 + g A^i \partial_i \varphi - \frac{1}{2} g^2 A^i A_i + \eta^1 \mathcal{P}_2 \Big). \end{aligned} \tag{86}$$

On account of these expressions, we deduce that the deformation modifies only the secondary constraint like

$$\gamma'_2 \equiv -\partial^i \pi_i + g\pi \approx 0, \tag{87}$$

while the primary one (see the former relation in (17)) is unchanged. In addition, our procedure preserves the Abelianity of the new constraints. The associated deformed first-class Hamiltonian is

$$\begin{aligned} H' = \int d^3x \Big(& \frac{1}{2} \pi_i \pi_i + \frac{1}{4} F_{ij} F^{ij} - A^0 (\partial^i \pi_i - g\pi) \\ & + \frac{1}{2} \pi^2 - \frac{1}{2} (\partial_i \varphi - g A_i)(\partial^i \varphi - g A^i) \Big). \end{aligned} \tag{88}$$

By passing to the Lagrangian version of the resulting coupled theory, we find the action

$$\begin{aligned} S'^{L}[\varphi, \bar{\varphi}, A^\mu] = \int d^4x \Big(& -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ & + \frac{1}{2} (\partial_\mu \varphi - g A_\mu)(\partial^\mu \varphi - g A^\mu) \Big), \end{aligned} \tag{89}$$

invariant under the gauge transformations

$$\delta_\epsilon A^\mu = \partial^\mu \epsilon, \quad \delta_\epsilon \varphi = g\epsilon. \tag{90}$$

Thus, the gauge symmetry of the real scalar field in the framework of the deformed system can again be deduced by performing the gauging of the corresponding global shift symmetry (63), present at the level of the starting free model. Analyzing the coupling between the real scalar field and the Abelian gauge field emphasized by our deformation procedure, we conclude that it is precisely a Stuckelberg-like coupling between a zero- and a one-form.

4 Couplings between an Abelian gauge field and a Dirac field

Here, we derive the consistent Hamiltonian interactions between an Abelian gauge field and a Dirac field, $(\psi^\alpha, \bar{\psi}_\alpha)$. The starting point is a free Lagrangian action that is equal to the sum of the actions of an Abelian gauge field and a Dirac field

$$\begin{aligned} & \tilde{S}_0^{L}[\psi^\alpha, \bar{\psi}_\alpha, A^\mu] \\ & = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}_\alpha (i(\gamma^\mu)^\alpha{}_\beta \partial_\mu - m\delta^\alpha{}_\beta) \psi^\beta \right), \end{aligned} \tag{91}$$

where the spinor fields are fermionic, and γ^μ is the standard notation for Dirac's gamma matrices. The bar operation now signifies Dirac conjugation. The action (91) is known to be invariant under the (bosonic) rigid one-parameter symmetry

$$\Delta \psi^\alpha = i\psi^\alpha \xi, \quad \Delta \bar{\psi}_\alpha = -i\bar{\psi}_\alpha \xi, \tag{92}$$

involving only the spinors, that gives, according to Noether's theorem, the conservation law

$$\partial_\mu j^\mu = i \frac{\delta^R \mathcal{L}^{(D)}}{\delta \psi^\alpha} \psi^\alpha - i \frac{\delta^R \mathcal{L}^{(D)}}{\delta \bar{\psi}_\alpha} \bar{\psi}_\alpha, \tag{93}$$

which emphasizes the conserved current

$$j^\mu = \bar{\psi}_\alpha (\gamma^\mu)^\alpha{}_\beta \psi^\beta, \tag{94}$$

where

$$\frac{\delta^R \mathcal{L}^{(D)}}{\delta \psi^\alpha} = -(i(\gamma^\mu)^\beta{}_\alpha \partial_\mu + m\delta^\beta{}_\alpha) \bar{\psi}_\beta, \tag{95}$$

$$\frac{\delta^R \mathcal{L}^{(D)}}{\delta \bar{\psi}_\alpha} = -(i(\gamma^\mu)^\alpha{}_\beta \partial_\mu - m\delta^\alpha{}_\beta) \psi^\beta, \tag{96}$$

and $\mathcal{L}^{(D)}$ obviously denotes the Dirac Lagrangian. The upper index R (L) signifies the right (left) derivative.

From the canonical analysis of this model we extract the constraints and the canonical Hamiltonian

$$G_1 \equiv \pi_0 \approx 0, \quad G_2 \equiv -\partial^i \pi_i \approx 0, \tag{97}$$

$$\chi_\alpha \equiv \Pi_\alpha - \frac{\mathrm{i}}{2}(\gamma^0)^\beta{}_\alpha \bar{\psi}_\beta \approx 0,$$
$$\bar{\chi}^\alpha \equiv \bar{\Pi}^\alpha - \frac{\mathrm{i}}{2}(\gamma^0)^\alpha{}_\beta \psi^\beta \approx 0, \tag{98}$$

$$\tilde{H}_0 = \int \mathrm{d}^3x \left(\frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F^{ij} - A^0\partial^i\pi_i - \bar{\psi}_\alpha (\mathrm{i}(\gamma^i)^\alpha{}_\beta \partial_i - m\delta^\alpha{}_\beta)\psi^\beta \right). \tag{99}$$

In (98) and (99), $\bar{\Pi}^\alpha$ and Π_α denote the canonical momenta respectively conjugated to the fields $\bar{\psi}_\alpha$ and ψ^α. The constraints (97) are first class and Abelian, while those of (98) are second class. Eliminating the second-class constraints by means of the Dirac bracket $[,]$ constructed with respect to themselves, we find that the spinors ψ^α and $\bar{\psi}_\alpha$ become conjugated

$$[\psi^\alpha, \bar{\psi}_\beta] = \mathrm{i}(\gamma^0)^\alpha{}_\beta, \tag{100}$$

the resulting theory evolving on a reduced phase space described by the fields/momenta (A^μ, π_μ), $(\psi^\alpha, \bar{\psi}_\alpha)$ and displaying only the Abelian first-class constraints (97), together with the first-class Hamiltonian (99). Related to the Hamiltonian BRST symmetry associated with this free theory, we mention that our discussion from Sect. 3.1 remains valid in the Abelian gauge field sector, with the exception of the bracket, which should be interpreted as Dirac instead of Poisson. Thus, all formulas (19)–(28) connected with this sector will be used in the sequel, while the ones describing the complex scalar component should be removed and replaced by

$$\mathrm{antigh}(\psi^\alpha) = \mathrm{antigh}(\bar{\psi}_\alpha) = 0,\ \mathrm{pgh}(\psi^\alpha) = \mathrm{pgh}(\bar{\psi}_\alpha) = 0, \tag{101}$$

$$\delta\psi^\alpha = 0,\ \delta\bar{\psi}_\alpha = 0,\ \gamma\psi^\alpha = 0,\ \gamma\bar{\psi}_\alpha = 0. \tag{102}$$

With these observations at hand, we next proceed to analyzing the Hamiltonian deformation procedure.

The analysis of (4), (5), etc., correlated with the deformation of the BRST charge (19) goes along exactly the same line as employed for the complex or real scalar field, and allows us to write down the deformed solution in the form

$$\Omega = \int \mathrm{d}^3x (\pi_0\eta^1 - (\partial^i\pi_i - g\tilde{\mu}_0(\psi^\alpha, \bar{\psi}_\alpha))\eta^2), \tag{103}$$

where the (so far) unknown bosonic function $\tilde{\mu}_0(\psi^\alpha, \bar{\psi}_\alpha)$ depends only on the spinor fields, is bosonic, and satisfies the properties $\mathrm{antigh}(\tilde{\mu}_0) = 0$, $\mathrm{pgh}(\tilde{\mu}_0) = 0$ and $\gamma\tilde{\mu}_0 = 0$. Thus, the only non-vanishing piece in the deformed BRST charge is the one corresponding to the first order in the deformation parameter,

$$\Omega_1 = \int \mathrm{d}^3x\, \tilde{\mu}_0(\psi^\alpha, \bar{\psi}_\alpha)\eta^2. \tag{104}$$

The unknown function will be found during the identification of the deformed BRST-invariant Hamiltonian, governed by (9), (10), etc.

As the BRST-invariant Hamiltonian of the free system under study is (34), with H_0 substituted with $\tilde{H}_0$, from (104) it follows that

$$\begin{aligned}[H_{0\mathrm{B}}, \Omega_1] = &-\int \mathrm{d}^3x (\mathrm{i}(\mathrm{i}(\gamma^j)^\beta{}_\alpha \partial_j\bar{\psi}_\beta + m\bar{\psi}_\alpha)(\gamma^0)^\alpha{}_\rho \bar{w}^\rho\eta^2 \\ &+ \mathrm{i}(\mathrm{i}(\gamma^j)^\beta{}_\alpha \partial_j\psi^\alpha - m\psi^\beta)(\gamma^0)^\rho{}_\beta w_\rho\eta^2 + \eta^1\tilde{\mu}_0) \\ = &\int \mathrm{d}^3x\, \tilde{\lambda},\end{aligned} \tag{105}$$

hence (9) reduces in the local form to

$$sh_1 + \tilde{\lambda} = \partial^i\tilde{n}_i, \tag{106}$$

for some $\tilde{n}_i$. In (105) we performed the notations $\bar{w}^\rho(x) = \int \mathrm{d}^3y(\delta^{\mathrm{L}}\tilde{\mu}_0(x^0, \boldsymbol{y})/\delta\bar{\psi}_\rho(x))$ and $w_\rho(x) = \int \mathrm{d}^3y(\delta^{\mathrm{L}}\tilde{\mu}_0(x^0, \boldsymbol{y})/\delta\psi^\rho(x))$. In order to remove the term linear in η^1 from the left-hand side of (106), we act like in the case of the complex or real scalar field, namely, we demand that

$$h_1 = \tilde{\mu}_0 A^0 + \tilde{\alpha}, \tag{107}$$

where the bosonic function $\tilde{\alpha}$ is unknown, and can depend only on ψ^α, $\bar{\psi}_\alpha$ and A^i. The dependence on A^i is required for ensuring the appearance of spatial derivatives via the Dirac bracket between H_1 and Ω_0, and, meanwhile, $\tilde{\alpha}$ should involve no ghosts or antighosts in order to prevent the existence of terms in $[H_1, \Omega_0]$ different from those in $[H_{0\mathrm{B}}, \Omega_1]$, which can be attained via a dependence also on ψ^α and $\bar{\psi}_\alpha$. Accordingly, we find

$$[H_1, \Omega_0] = \int \mathrm{d}^3x (\eta^1\tilde{\mu}_0 + \tilde{a}_i\partial^i\eta^2), \tag{108}$$

where $\tilde{a}_i(x) = \int \mathrm{d}^3y(\delta\tilde{\alpha}(x^0, \boldsymbol{y})/\delta A^i(x))$. From (105) and (108), we see that (106) becomes

$$\begin{aligned}&-\mathrm{i}((\mathrm{i}(\gamma^j)^\beta{}_\alpha \partial_j\bar{\psi}_\beta + m\bar{\psi}_\alpha)(\gamma^0)^\alpha{}_\rho \bar{w}^\rho \\ &+ (\mathrm{i}(\gamma^j)^\beta{}_\alpha \partial_j\psi^\alpha - m\psi^\beta)(\gamma^0)^\rho{}_\beta w_\rho)\eta^2 + \tilde{a}_i\partial^i\eta^2 \\ &= \partial^i\tilde{n}_i.\end{aligned} \tag{109}$$

The left-hand side of (109) reduces to a total derivative if

$$\begin{aligned}&-\mathrm{i}((\mathrm{i}(\gamma^j)^\beta{}_\alpha \partial_j\bar{\psi}_\beta + m\bar{\psi}_\alpha)(\gamma^0)^\alpha{}_\rho \bar{w}^\rho \\ &+ (\mathrm{i}(\gamma^j)^\beta{}_\alpha \partial_j\psi^\alpha - m\psi^\beta)(\gamma^0)^\rho{}_\beta w_\rho) \\ &= \partial^i\tilde{a}_i.\end{aligned} \tag{110}$$

Adding to both sides of (110) the term $-(\mathrm{i}(\gamma^0)^\beta{}_\alpha \partial_0\bar{\psi}_\beta)\, \mathrm{i}(\gamma^0)^\alpha{}_\rho \bar{w}^\rho$, as well as the quantity $-\mathrm{i}((\gamma^0)^\beta{}_\alpha \partial_0\psi^\alpha)\, \mathrm{i}(\gamma^0)^\rho{}_\beta w_\rho$, we deduce

$$\begin{aligned}&\frac{\delta^{\mathrm{R}}\mathcal{L}^{(D)}}{\delta\psi^\alpha}\mathrm{i}(\gamma^0)^\alpha{}_\rho \bar{w}^\rho + \frac{\delta^{\mathrm{R}}\mathcal{L}^{(D)}}{\delta\bar{\psi}_\alpha}\mathrm{i}(\gamma^0)^\rho{}_\alpha w_\rho \\ &= (\partial_0\bar{\psi}_\alpha)\bar{w}^\alpha + (\partial_0\psi^\alpha)w_\alpha + \partial^i\tilde{a}_i.\end{aligned} \tag{111}$$

Analyzing the structure of the last formula and replacing $\bar{w}^\rho$, w_ρ in terms of $\tilde{\mu}_0$, it turns out that its left-hand side gives the variation of the Dirac Lagrangian under the rigid one-parameter transformations

$$\Delta\psi^\alpha(x) = \mathrm{i}(\gamma^0)^\alpha{}_\rho \int \mathrm{d}^3y \frac{\delta^{\mathrm{L}}\tilde{\mu}_0(x^0,\boldsymbol{y})}{\delta\bar{\psi}_\rho(x)}\xi, \quad (112)$$

$$\Delta\bar{\psi}_\alpha(x) = \mathrm{i}(\gamma^0)^\rho{}_\alpha \int \mathrm{d}^3y \frac{\delta^{\mathrm{L}}\tilde{\mu}_0(x^0,\boldsymbol{y})}{\delta\psi^\rho(x)}\xi. \quad (113)$$

Identifying (112) and (113) with the well-known global one-parameter invariance (92) of Dirac theory, we are led to the equations

$$(\gamma^0)^\alpha{}_\rho \int \mathrm{d}^3y \frac{\delta^{\mathrm{L}}\tilde{\mu}_0(x^0,\boldsymbol{y})}{\delta\bar{\psi}_\rho(x)} = \psi^\alpha(x), \quad (114)$$

$$(\gamma^0)^\rho{}_\alpha \int \mathrm{d}^3y \frac{\delta^{\mathrm{L}}\tilde{\mu}_0(x^0,\boldsymbol{y})}{\delta\psi^\rho(x)} = -\bar{\psi}_\alpha(x), \quad (115)$$

that yield the solution

$$\tilde{\mu}_0(y) = \bar{\psi}_\alpha(y)(\gamma^0)^\alpha{}_\beta\psi^\beta(y). \quad (116)$$

Substituting (116) in (111) and using (93), we are provided with the equations $\int \mathrm{d}^3y(\delta\tilde{\alpha}(x^0,\boldsymbol{y})/\delta A_i(x)) = \bar{\psi}_\alpha(x)(\gamma^i)^\alpha{}_\beta\psi^\beta(x)$, which produce

$$\tilde{\alpha}(y) = \bar{\psi}_\alpha(y)(\gamma^i)^\alpha{}_\beta\psi^\beta(y)A_i(y). \quad (117)$$

Consequently, we have generated the first-order deformed BRST-invariant Hamiltonian:

$$H_1 = \int \mathrm{d}^3x(\bar{\psi}_\alpha(\gamma^0)^\alpha{}_\beta\psi^\beta A_0 + \bar{\psi}_\alpha(\gamma^i)^\alpha{}_\beta\psi^\beta A_i). \quad (118)$$

Further, let us study the higher-order deformations. By direct computation we get $[H_1, \Omega_1] = 0$, which combined with $\Omega_2 = 0$ allows us to take the solution of (10) to be $H_2 = 0$. Then, it is simple to check that we can choose

$$H_3 = H_4 = \cdots = 0. \quad (119)$$

In conclusion, the complete deformed BRST charge and BRST-invariant Hamiltonian that govern the couplings between an Abelian gauge field and a Dirac field are given by

$$\Omega = \int \mathrm{d}^3x(\pi_0\eta^1 - (\partial^i\pi_i - g\bar{\psi}_\alpha(\gamma^0)^\alpha{}_\beta\psi^\beta)\eta^2), \quad (120)$$

respectively,

$$H_{\mathrm{B}} = \int \mathrm{d}^3x\left(\frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F^{ij} - A^0\partial^i\pi_i - \bar{\psi}_\alpha(i(\gamma^i)^\alpha{}_\beta\partial_i - m\delta^\alpha{}_\beta)\psi^\beta + g\bar{\psi}_\alpha(\gamma^\mu)^\alpha{}_\beta\psi^\beta A_\mu + \eta^1\mathcal{P}_2\right). \quad (121)$$

Like in the case of the scalar field theory, from the above quantities we read off that the classical Hamiltonian interacting theory is subject to the deformed Abelian first-class constraints

$$\tilde{\gamma}_2 \equiv -\partial^i\pi_i + g\bar{\psi}_\alpha(\gamma^0)^\alpha{}_\beta\psi^\beta \approx 0, \quad (122)$$

and the former constraint in (97), as well as that the first-class Hamiltonian with respect to these constraints has the expression

$$\tilde{H} = \int \mathrm{d}^3x\left(\frac{1}{2}\pi_i\pi_i + \frac{1}{4}F_{ij}F^{ij} - A^0(\partial^i\pi_i - g\bar{\psi}_\alpha(\gamma^0)^\alpha{}_\beta\psi^\beta) - \bar{\psi}_\alpha(\mathrm{i}(\gamma^i)^\alpha{}_\beta\partial_i - m\delta^\alpha{}_\beta)\psi^\beta + g\bar{\psi}_\alpha(\gamma^\mu)^\alpha{}_\beta\psi^\beta A_\mu\right), \quad (123)$$

where the first-class behavior is considered in terms of the Dirac bracket (100). If we take the necessary steps to the Lagrangian framework, we discover that the resulting interacting theory displays the Lagrangian action

$$\tilde{S}^{\mathrm{L}}[\psi^\alpha, \bar{\psi}_\alpha, A^\mu] = \int \mathrm{d}^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_\alpha(\mathrm{i}(\gamma^\mu)^\alpha{}_\beta D_\mu - m\delta^\alpha{}_\beta)\psi^\beta\right), \quad (124)$$

invariant under the gauge transformations

$$\delta_\epsilon A^\mu = \partial^\mu\epsilon, \quad \delta_\epsilon\psi^\alpha = \mathrm{i}g\psi^\alpha\epsilon, \quad \delta_\epsilon\bar{\psi}_\alpha = -\mathrm{i}g\bar{\psi}_\alpha\epsilon, \quad (125)$$

where the covariant derivative D_μ takes the form (61). We observe that, exactly like for the complex or real scalar field, the spinors bear now some gauge invariances, resulting from the original rigid ones in a direct manner by gauging. An interesting difference between this model and the scalar theory is that while there we have obtained non-trivial pieces for the BRST-invariant Hamiltonian at order two in the deformation parameter, the similar quantity stops here at order one. This feature is essentially due to the statistics of the present matter fields, which are spinors, hence fermionic. Thus, we can conclude that as a result of our deformation scheme we obtained the well-known model describing the coupling between the electromagnetic and spinor fields, namely spinor electrodynamics.

5 Conclusion

In conclusion, in this paper we have derived the consistent Hamiltonian interactions between an Abelian gauge field and the complex scalar field, the massless real scalar field, respectively, Dirac field. Our approach is based on the deformation of the BRST charge and BRST-invariant Hamiltonian associated with the uncoupled theories involving these fields. The derivation of the solutions to the main equations that govern our BRST deformation procedure essentially relies on the presence of some conserved currents corresponding to the rigid symmetries of the matter fields from the "free" models. The first-order deformations of both BRST charge and BRST-invariant Hamiltonian can be written in the form $\Omega_1 = \pm \int d^3x q \eta^2$, respectively, $H_1 = \pm \int d^3x (qA^0 + j_i A^i)$, in the case of all analyzed models, where q is the Hamiltonian charge density of the associated conserved currents. For the scalar case we see that $[H_1, \Omega_1]$ is non-vanishing due to the fact that $[j_i, q]$ is not zero, which requires non-trivial second-order deformations of the BRST-invariant Hamiltonian. In the case of the Dirac theory we have $[H_1, \Omega_1] = 0$, so the second-order deformations of H_{0B} can be taken to vanish. It is interesting to note that, apart from other situations [41–44], where the deformation of the BRST charge can be computed in a self-consistent manner, here we need to alternate it with the deformation of the BRST-invariant Hamiltonian in order to reach some complete solutions. As a result of our method we discover scalar electrodynamics, a Stuckelberg-like coupling, respectively, spinor electrodynamics. All the couplings are local, and the matter fields bear some gauge invariances that can be produced via the gauging of the original global symmetries. As expected, the $U(1)$ gauge invariance of Maxwell's field is kept unchanged for all models during the deformation process.

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